SKEE1223: Digital Electronics

5 – Logic Gates and Boolean Algebra

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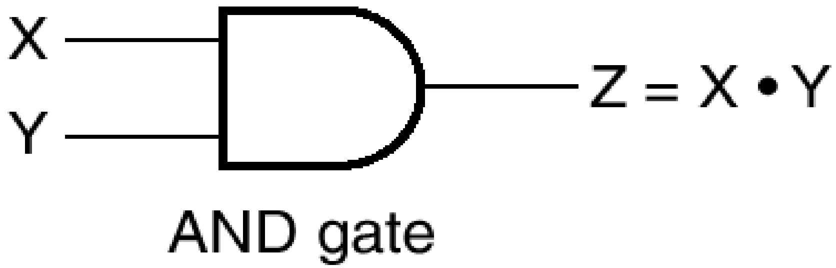
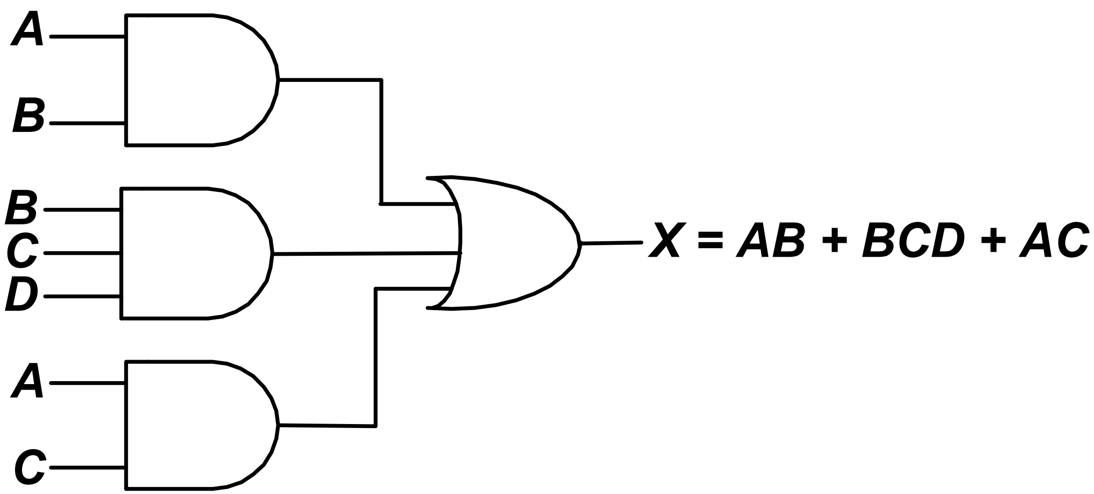
Logic Gates and Boolean Algebra

- Logic Gates
  - Inverter, OR, AND, Buffer, NOR, NAND, XOR, XNOR
- Boolean Theorem
  - Commutative, Associative, Distributive Laws
  - Basic Rules
- DeMorgan's Theorem
- Universal Gates
  - NAND and NOR
- Canonical/Standard Forms of Logic
  - Sum of Product (SOP)
  - Product of Sum (POS)
  - Minterm and Maxterm

Standard Forms of Boolean Expressions

- Boolean expressions





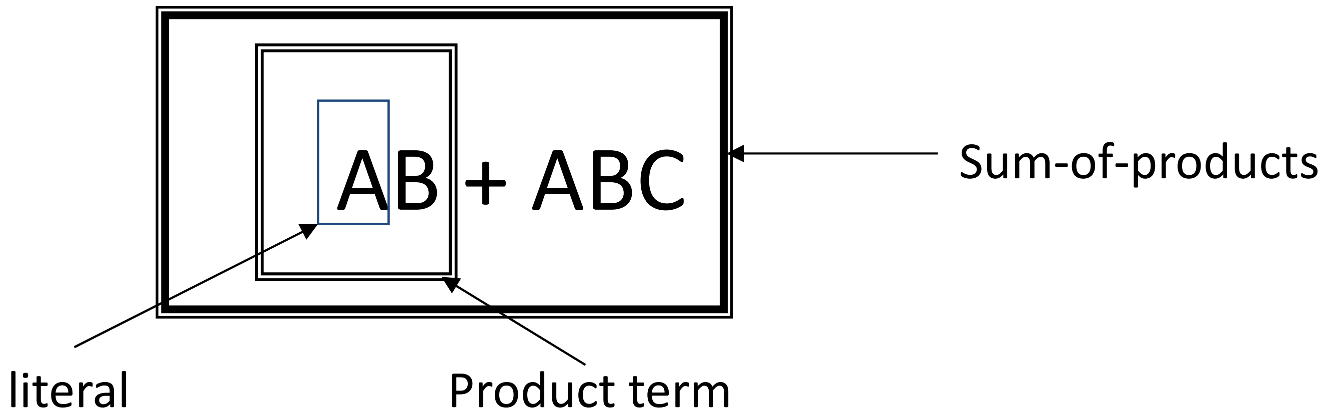
Standard forms of Boolean Expressions

- Two standard forms
  - Sum-of-products (SOP) form
  - Product-of-sums (POS) form
- Standardization
  - Evaluation, simplification and implementation of boolean expressions become more systematic, easy.

Sum-of-Products (SOP)

- Product term—a term that consists of the product of literals
- Sum-of-products—sum of two or more product terms

Example 1: $A\overline{B}C + BC + AC$



Sum of Product (SOP)

- Boolean expressions are expressed as the sum of product, example:

minterm

$$ABC + CDE + \overline{B}C\overline{D}$$

literal

- Each variable or their complements is called *literals*
- Each product term is called *minterm*

Sum-of-Products (SOP)

- $A+BC$
- $A\overline{B}+C\overline{D}$
- $A\overline{B}\,\overline{C}+\overline{B}D$
- $ABC+BD$

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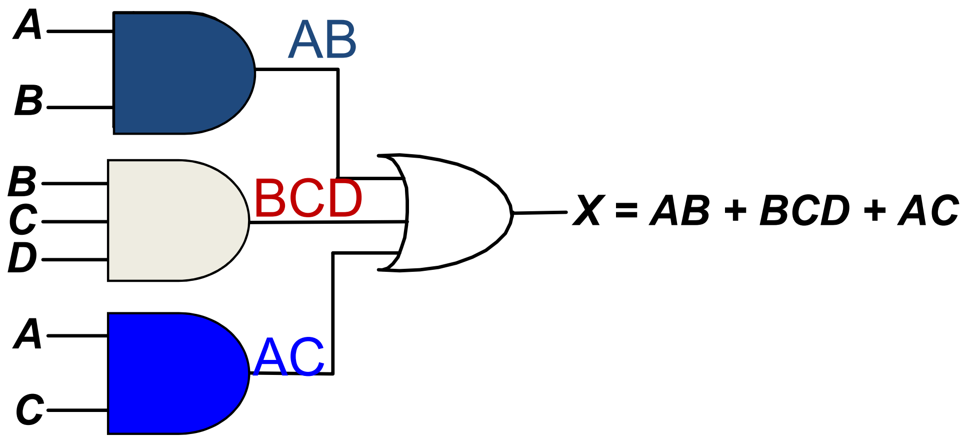
Domain of a SOP expression

- $A\overline{B}+A\overline{BC}$
  - Domain: A, B, C
- $ABC+CD\overline{E}+B\overline{C}\,\overline{D}$
  - Domain: A, B, C, D, E
- $X+Y+XYZ$

AND/OR Implementation of an SOP expression

- X=AB+BCD+AC



- Y=A + BD + BC + CD

Conversion of a general expressions to SOP form

- $A(B+CD) = AB + ACD$
- $(A+C)(AB+AC)=?$
- $AB + B(CD+EF)=?$
- $\overline{\overline{(A+B)}+C}$

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Canonical/Standard SOP Form

- SOP expressions
  - $\overline{A}BC+ABD+AB\overline{C}D$
  - Domain ?
- Standard SOP expression—an **expression** where **all** the **variables** in the **domain** appear in **each** of its **product term**.
- Examples: $\overline{A}\,\overline{B}\,\overline{C}+\overline{A}BC+\overline{A}BC$

  $\overline{X}YZ+XY\overline{Z}$

SOP (cont.)

- In SOP, a single overbar cannot extend over more than one variable, example:

$$AB + A\overline{BC}$$ ← Not SOP because $\overline{\text{BC}}$

- *Standard SOP* forms must contain all of the variables in the domain of the expression for each product term, example:

$$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

SOP (cont.)

- In the following SOP form,

$$A\overline{B}C + \overline{A}\overline{B} + AB\overline{C}D$$

  - How many minterms are there? => 3
  - How many literals in the second product term? => 2
  - Is it in a standard SOP form? => No
  - How do we convert the boolean expression to standard SOP form?

Conversion to Canonical/Standard SOP

Example: A+B

- **Step 1: Multiply each non-standard term with 1.**

$$A + B = A \bullet 1 + B \bullet 1$$

- **Step 2: Replace 1 with sum of the missing variable and its complement**

$$A + B = A(B + \overline{B}) + B(A + \overline{A})$$

- **Ans:**

$$A + B = AB + A\overline{B} + AB + \overline{A}B$$

$$= AB + A\overline{B} + \overline{A}B$$

SOP (cont.)

- To convert SOP to its standard form, we use the boolean rules

  $A + \overline{A} = 1$

  $A(B + C) = AB + AC$

- We have

  $$A\overline{B}C + \overline{A}\overline{B} + AB\overline{C}D$$

- The first product term is missing the variable D, and the second product term is missing C and D

SOP (cont.)

$$A\overline{B}C + \overline{A}\,\overline{B} + AB\overline{C}D$$

Apply $D + \overline{D} = 1$ and $C + \overline{C} = 1$

$$= A\overline{B}C(D + \overline{D}) + \overline{A}\,\overline{B}(C + \overline{C})(D + \overline{D}) + AB\overline{C}D$$

Apply the distributive law

$$= A\overline{B}CD + A\overline{B}C\overline{D} + (\overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C})(D + \overline{D}) + AB\overline{C}D$$

$$= A\overline{B}CD + A\overline{B}C\overline{D} + \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}C\overline{D} +$$

$$\overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + AB\overline{C}D$$ ← Standard SOP form

Exercise

$$a. \quad \overline{A}BC + A\overline{B} + \overline{A}BC\overline{D}$$

$$b. \quad \overline{W}XY + \overline{W}Y\overline{Z} + WX\overline{Y}$$

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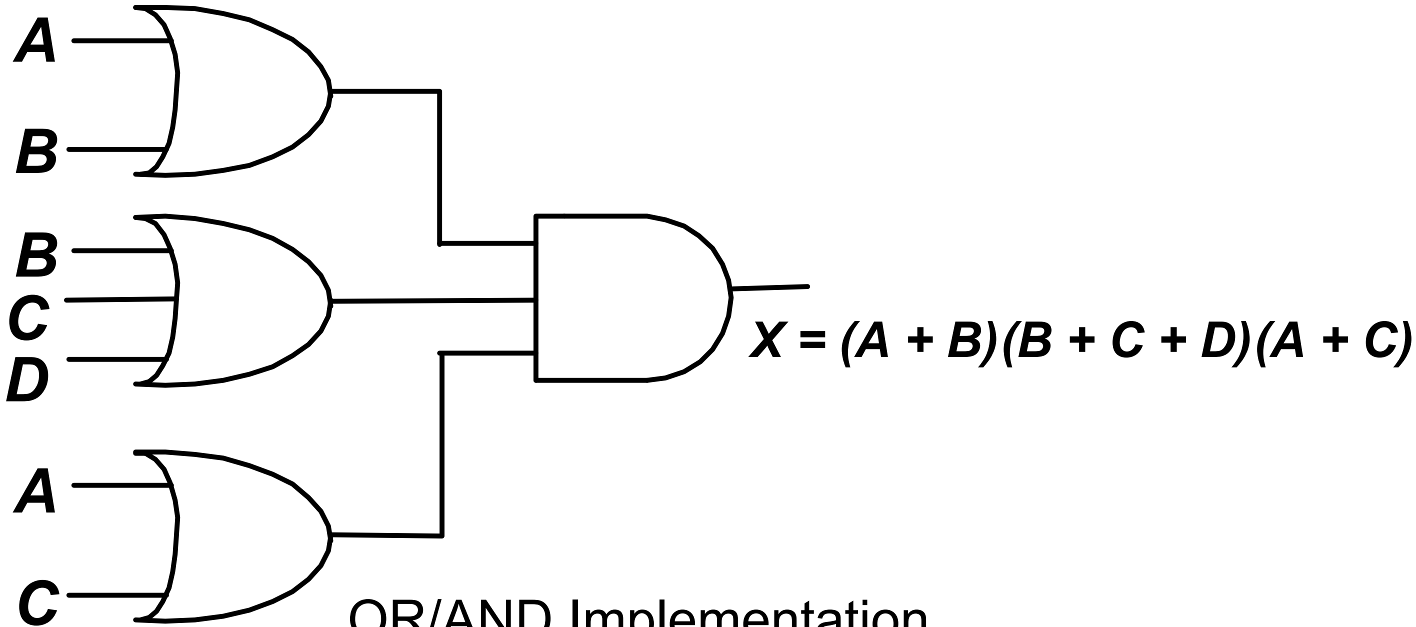
Product-of-Sums (POS)

- Sum term—a term that consists of the sum of literals
- Product-of-sums—multiplication of two or more sum terms.
- Example 1: $(A + B)(\overline{A} + B + C)$
- Example 2: Sum term?

$\overline{A}+\overline{B}+\overline{C}$ ☑ $\overline{A+B+C}$ ☒

Product-of-Sums (POS)

- Implementation of the POS expression
- (A + B) (B + C + D) (A + C)



OR/AND Implementation

Product of Sum (POS)

- Boolean expressions are expressed as the product of sum, example:

$$(\overline{A}+B)(A+\overline{B}+C)$$

literal

maxterm

Domain of a POS expression

- (A+B+C)(C+D+E)(B+C+D)
  - Domain: A, B, C, D, E
- (A+B)(A+B+C)(C+D)
  - Domain: A, B, C, D
- (U+T)(M+F+K)(T+E)(M)
  - Domain:

POS (cont.)

- In POS, a single overbar cannot extend over more than one variable, example:

$$(\overline{A}+B)(A+\overline{B+C})$$ ← Not SOP because $\overline{\text{B+C}}$

- *Standard POS* forms must contain all of the variables in the domain of the expression for each sum term, example:

$$(A+B+C)(A+B+\overline{C})(A+\overline{B}+C)$$

Canonical/Standard POS Expression

- An expression where all of the variables in the domain appear in each of its sum term.
- (A+B+D)(A+B+C) is not a standard POS expression.
- Examples:
- $(X+Y+Z)(X+Y+\overline{Z})$
- $(\overline{U}+T+\overline{M})(U+\overline{T}+M)$

POS (cont.)

- In the following POS form,

$$(A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

  – Is it in a standard POS form? => No
  – How do we convert the boolean expression to standard POS form?

Conversion to Canonical/Standard POS

Example: $(W+X+Y)(X+Y+Z)$

- **Step 1: Add 0 to each nonstandard product term.**

$$(W+X+Y+0)(X+Y+Z+0)$$

- **Step 2: Replace 0 by a product of the missing variable and its complement.**

$$(W+X+Y+Z\overline{Z})(X+Y+Z+W\overline{W})$$

- **Step 3: Apply rule 12: A+BC=(A+B)(A+C)**

$$(W+X+Y+Z)(W+X+Y+\overline{Z})(X+Y+Z+W)(X+Y+Z+\overline{W})$$

POS (cont.)

- To convert POS to its standard form, we use the boolean rules

  $A \cdot \overline{A} = 0$

  $A + BC = (A + B)(A + C)$

- We have

  $$(A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

- The first sum term is missing the variable D, and the second sum term is missing A

POS (cont.)

$$(A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

Apply D.$\overline{\text{D}}$ = 0 and A.$\overline{\text{A}}$ = 0 to first and second terms

$$(A+\overline{B}+C+D.\overline{D})(A.\overline{A}+\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

Expand first and second terms

$$(A+\overline{B}+C+D)(A+\overline{B}+C+\overline{D})(A+\overline{B}+C+\overline{D})(\overline{A}+\overline{B}+C+\overline{D})$$
$$(A+\overline{B}+\overline{C}+D)$$

Standard POS form

Exercise

$$a. \quad (\overline{A}+B+C)(A+\overline{B})(B+C)$$

$$b. \quad (\overline{P}+Q)(\overline{P}+\overline{R}+S)$$

Truth Table

- A common way of representing a logical operation of a circuit.
- If *n* is number of inputs, combinations in a truth table equals 2^n.
  - e.g. inputs is A,B,C, there will be $2^3=8$ combinations

A	B	C	Output
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

SOP to Truth Tables

- Develop a truth table for the SOP expression

$$\bar{A}\bar{B}C + A\bar{B}\bar{C} + ABC$$

- Domain = A, B, C : combinations = 2^3 = 8
- What binary value makes the product term = 1?

$\bar{A}\bar{B}C$ 0 0 1	= 1 1 1 = 1
$A\bar{B}\bar{C}$ 1 0 0	= 1 1 1 = 1
ABC 1 1 1	= 1 1 1 = 1

INPUTS			OUTPUT	PRODUCT TERM
A	B	C	X	
0	0	0	0	
0	0	1	1	$\bar{A}\bar{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\bar{B}\bar{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

- Fill the truth table

POS to Truth Tables

- Develop a truth table for the POS expression

$$(A + B + C)(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + B + \overline{C})(\overline{A} + \overline{B} + C)$$

- Domain = A, B, C. combinations = $2^3 = 8$
- What binary value makes the sum term = 0?

(A+B+C)
0 0 0 = 0
(A+B'+C)
0 1 0 = 0 0 0 = 0
(A+B'+C')
0 1 1 = 0 0 0 = 0
(A'+B+C')
1 0 1 = 0 0 0 = 0
(A'+B'+C)
1 1 0 = 0 0 0 = 0

INPUTS A	B	C	OUTPUT X	PRODUCT TERM
0	0	0	0	$(A + B + C)$
0	0	1	1	
0	1	0	0	$(A + \overline{B} + C)$
0	1	1	0	$(A + \overline{B} + \overline{C})$
1	0	0	1	
1	0	1	0	$(\overline{A} + B + \overline{C})$
1	1	0	0	$(\overline{A} + \overline{B} + C)$
1	1	1	1	

- Fill the truth table

Boolean Expressions and Truth Tables

- Step 1: determine domain and combinations of binary values ➔ input
- Step 2: convert expression to Standard SOP/POS.
- Step 3: find the binary values that make the product
  - Equal to 1 for SOP, e.g ABC = 111
  - Equal to 0 for POS, e.g (A+B+C) = 000
- Step 4: fill the remaining blanks with inverse values

Homework

- Make a truth table for the following functions:

$$a. \quad F = AB + A\overline{C}$$

$$b. \quad F = (A + C)(B + \overline{C})$$

Truth Table to SOP Form

- Can write SOP form of equation directly from truth table.

A	B	C	F	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	← A'BC
1	0	0	1	← AB'C'
1	0	1	1	← AB'C
1	1	0	1	← ABC'
1	1	1	1	← ABC

Note that each term has ALL variables. If a product term has ALL variables present, it is a MINTERM.

F(A,B,C) = A'BC + AB'C' + AB'C + ABC' + ABC

Truth Table to SOP Form

- Can write POS form of equation directly from truth table.

A	B	C	F	
0	0	0	0	← A+B+C
0	0	1	0	← A+B+C’
0	1	0	0	← A+B’+C
0	1	1	1	
1	0	0	1	
1	0	1	1	
1	1	0	1	
1	1	1	1	

Note that each term has ALL variables. If a sum term has ALL variables present, it is a MAXTERM.

F(A,B,C) = (A+B+C) (A+B+C’)(A+B’+C)

Minterms and Maxterms

We saw that:

SOP form:

$$F(A, B, C) = \overline{A}BC + \overline{A}\,\overline{B}C + A\overline{B}C + AB\overline{C} + ABC$$

If a product term has all variables present, it is a **MINTERM.**

POS form:

$$F(A, B, C) = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)$$

If a sum term has all variables present, it is a **MAXTERM.**

All Boolean functions can be written in terms of either Minterms or Maxterms.

Using Minterms, Maxterms

A boolean function can be written in terms of Minterm or Maxterm notation as a shorthand method of specifying the function.

$$F(A,B,C) = A'BC + AB'C' + AB'C + ABC' + ABC' + ABC$$
$$= m_3 + m_4 + m_5 + m_6 + m_7$$
$$= \Sigma\, m(3,4,5,6,7)$$

$$F(A,B,C) = (A+B+C)\,(A+B+C')(A+B'+C)$$
$$= M_0\, M_1\, M_2$$
$$= \prod M(0,1,2)$$

Minterms correspond to '1's of F, Maxterms correspond to '0's of F in truth table.

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Minterms / Maxterms to Truth Table

Minterms correspond to '1's in Truth table

$F(A,B,C) = \Sigma\, m(1,2,6)$

$= m_1 + m_2 + m_6$

$= A'B'C + A'BC' + ABC'$

	A	B	C	F
M_0 →	0	0	0	0
m_1 →	0	0	1	1
m_2 →	0	1	0	1
M_3 →	0	1	1	0
M_4 →	1	0	0	0
M_5 →	1	0	1	0
m_6 →	1	1	0	1
M_7 →	1	1	1	0

Maxterms correspond to '0's in Truth table.

$F(A,B,C) = \Pi M(0,3,4,5,7)$

$= M_0 + M_3 + M_4 + M_5 + M_7$

$= (A+B+C)\ (A+B'+C')\ (A'+B+C)\ (A'+B+C')\ (A'+B'+C')$

Minterms / Maxterms to Truth Table

Each line in a truth table represents both a Minterm and a Maxterm.

Row No.	A B C	Minterms (SOP)	Maxterms (POS)
0	0 0 0	A'B'C' $= m_0$	A+B+C $= M_0$
1	0 0 1	A'B'C $= m_1$	A+B+C' $= M_1$
2	0 1 0	A'B C' $= m_2$	A+B'+C $= M_2$
3	0 1 1	A'B C $= m_3$	A+B'+C' $= M_3$
4	1 0 0	A B'C' $= m_4$	A'+B+C $= M_4$
5	1 0 1	A B'C $= m_5$	A'+B+C' $= M_5$
6	1 1 0	A B C' $= m_6$	A'+B'+C $= M_6$
7	1 1 1	A B C $= m_7$	A'+B'+C' $= M_7$

Minterm and Maxterm

- Minterm: Product terms in SOP
- Maxterm: Sum terms in POS
- Standard forms of SOP and POS can be derived from truth tables

A	B	C	Z	
0	0	0	0	$A+B+C$
0	0	1	1	$\overline{A}\overline{B}C$
0	1	0	0	$A+\overline{B}+C$
0	1	1	0	$A+\overline{B}+\overline{C}$
1	0	0	0	$\overline{A}+B+C$
1	0	1	1	$A\overline{B}C$
1	1	0	1	$AB\overline{C}$
1	1	1	1	ABC

For SOP form,

$$Z = \overline{A}\overline{B}C + A\overline{B}C + AB\overline{C} + ABC$$
$$= \sum m(1,5,6,7)$$

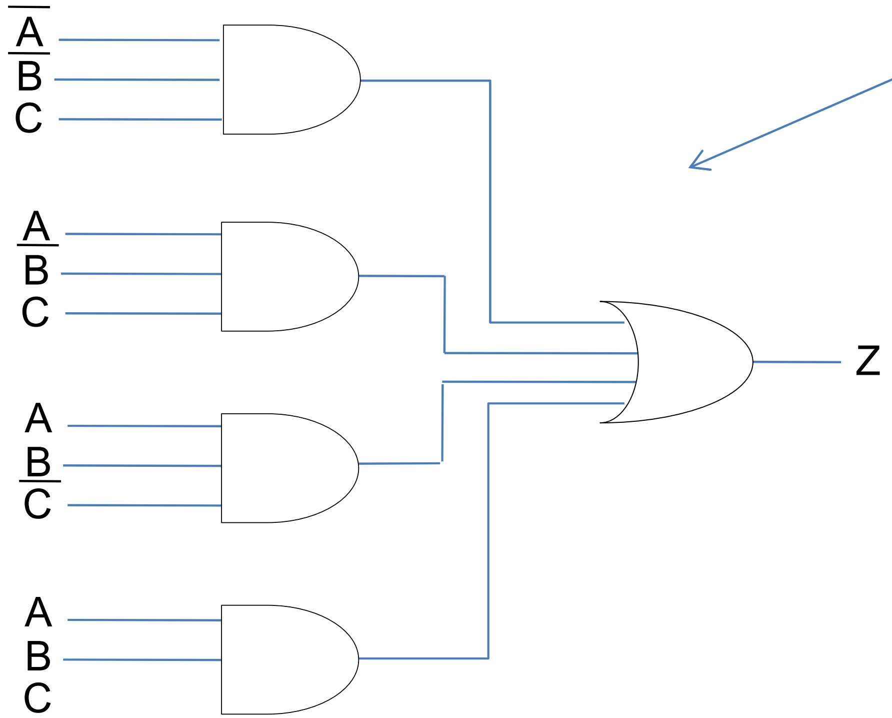
For POS form,

$$Z = (A+B+C)(A+\overline{B}+C)$$
$$(A+\overline{B}+\overline{C})(\overline{A}+B+C)$$
$$= \prod M(0,2,3,4)$$

Minterm and Maxterm

- How to design minterms – AND-OR logic

$$Z = \overline{A}\,\overline{B}C + A\overline{B}C + AB\overline{C} + ABC$$

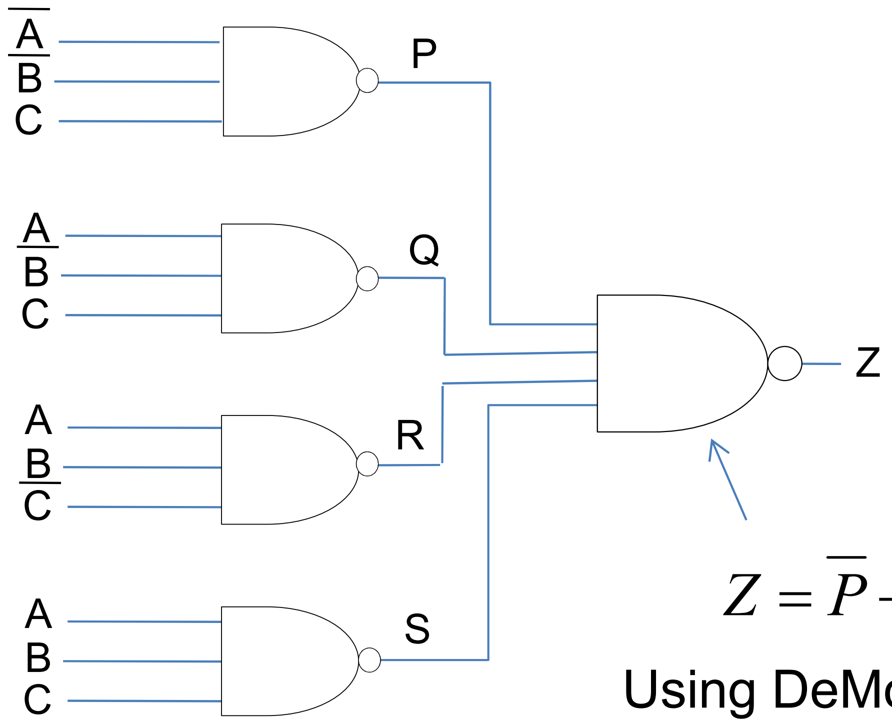


Also known as 2 level logic

Minterm and Maxterm

- How to design minterms – NAND-NAND Logic

$$Z = \overline{A}\,\overline{B}C + A\overline{B}C + AB\overline{C} + ABC$$



$$Z = \overline{P} + \overline{Q} + \overline{R} + \overline{S}$$

Using DeMorgan's Theorem

$$Z = \overline{P \cdot Q \cdot R \cdot S}$$

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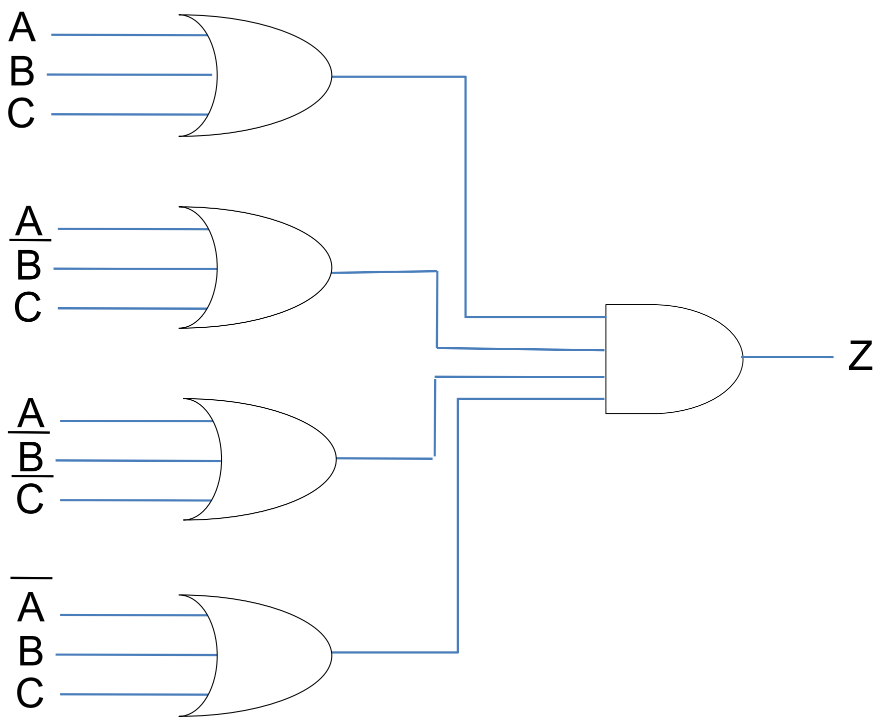
Minterm and Maxterm

- How to design maxterms – OR-AND Logic

$$Z = (A+B+C)(A+\overline{B}+C)(A+\overline{B}+\overline{C})(\overline{A}+B+C)$$



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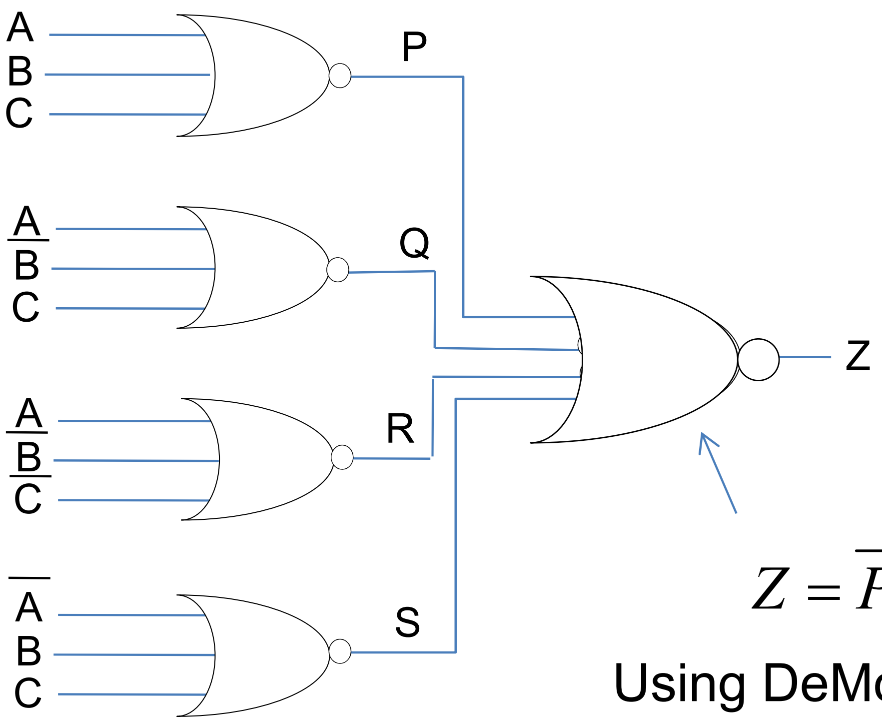
Minterm and Maxterm

- How to design maxterms – NOR-NOR Logic

$$Z = (A+B+C)(A+\overline{B}+C)(A+\overline{B}+\overline{C})(\overline{A}+B+C)$$



$$Z = \overline{P}\cdot\overline{Q}\cdot\overline{R}\cdot\overline{S}$$

Using DeMorgan's Theorem

$$Z = \overline{P+Q+R+S}$$

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Minterm and Maxterm

- Can the minterm and maxterm logic be optimized?
  - Yes, using Boolean algebra – explore yourself
  - Yes, using Karnaugh maps – next lecture

Conversion of Circuits

Example : $F(A,B,C) = A.\overline{B} + B.C + \overline{A}.\overline{C}$

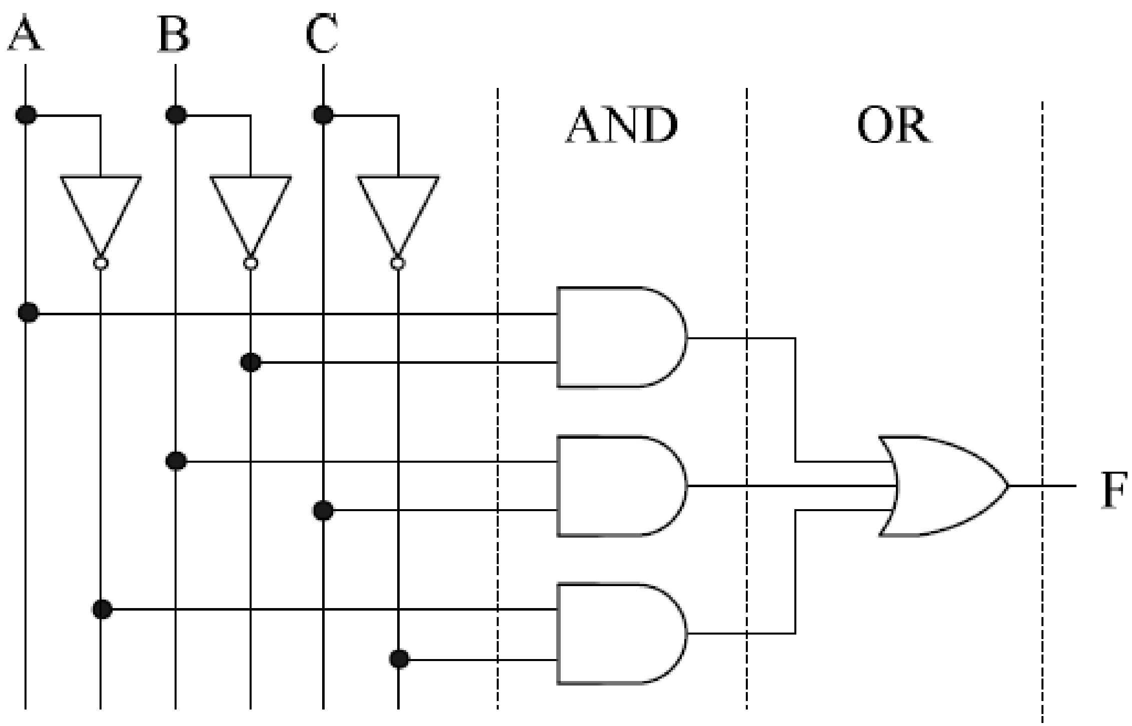
This SOP function can be implemented using:

i. AND-OR circuit

ii. NAND-NAND circuit

Conversion of Circuits

$$F(A, B, C) = A.\overline{B} + B.C + \overline{A}.\overline{C}$$



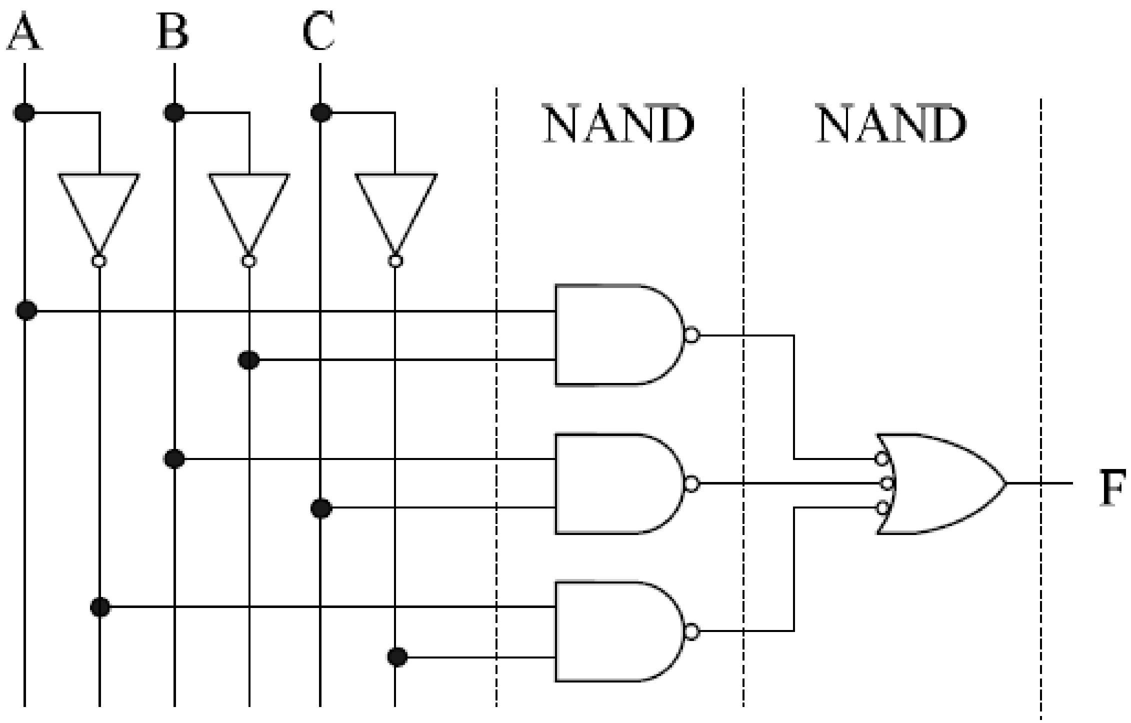
AND-OR circuit

Conversion of Circuits

$$F(A,B,C) = A.\bar{B} + B.C + \bar{A}.\bar{C} = \overline{\overline{A.\bar{B} + B.C + \bar{A}.\bar{C}}}$$

$$= \overline{\overline{(A.\bar{B})}.\overline{(B.C)}.\overline{(\bar{A}.\bar{C})}}$$

Use DeMorgan's Theorem



NAND-NAND circuit

Conversion of Circuits

Example: $F(A,B,C) = (\overline{A} + \overline{B} + C).(A + B + \overline{C})$

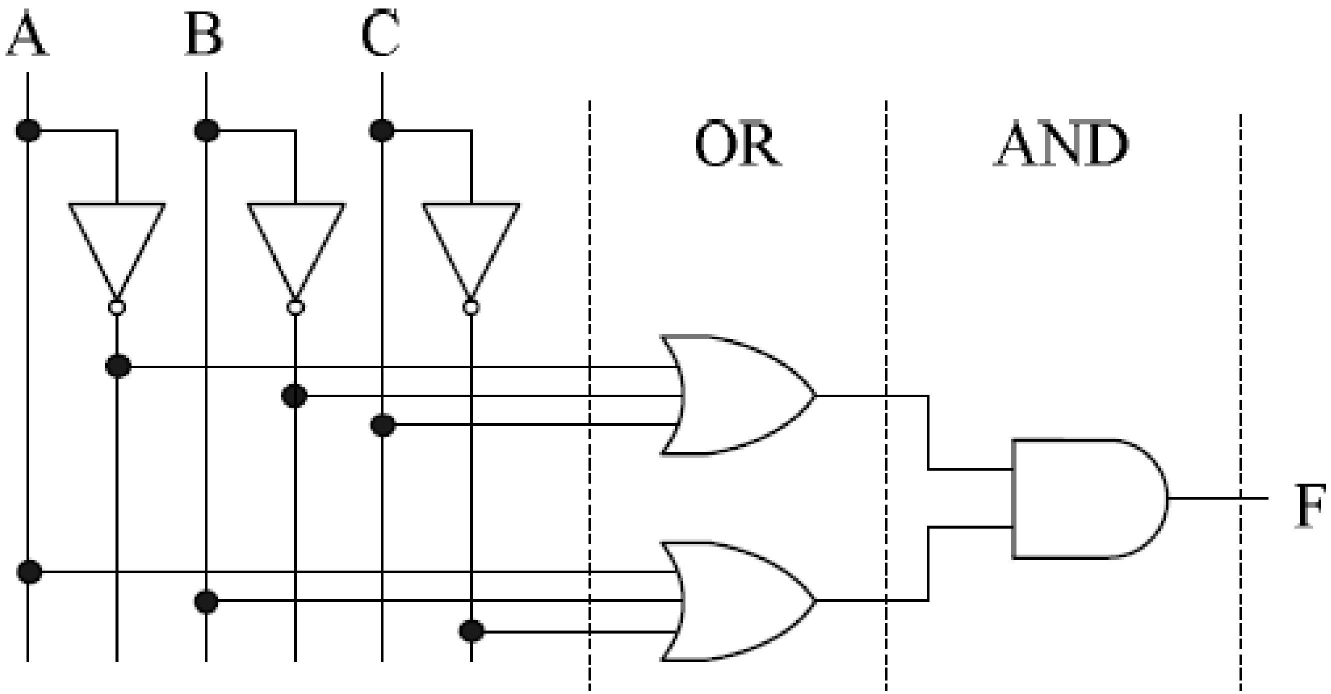
This POS function can be implemented using:

i. OR-AND circuit

ii. NOR-NOR circuit

Conversion of Circuits

$$F(A,B,C) = (\overline{A} + \overline{B} + C).(A + B + \overline{C})$$

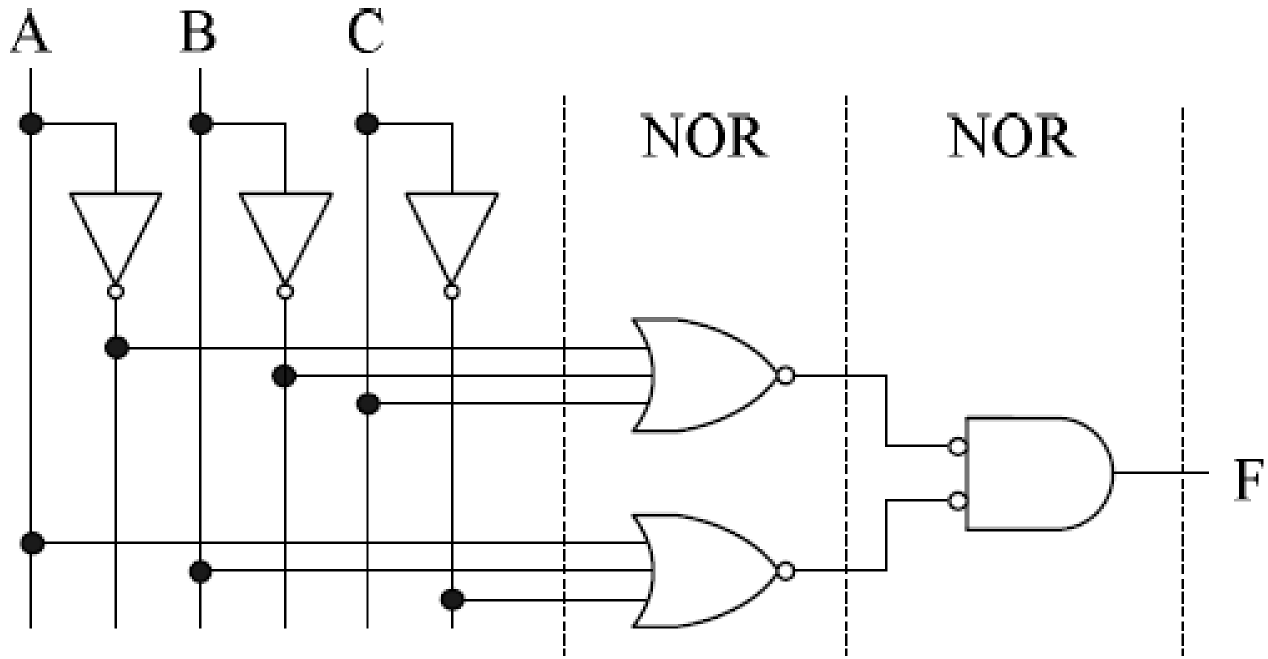


OR-AND circuit

Conversion of Circuits

$$F(A,B,C) = (\overline{A}+\overline{B}+C).(A+B+\overline{C}) = \overline{\overline{(\overline{A}+\overline{B}+C).(A+B+\overline{C})}}$$

$$= \overline{\overline{(\overline{A}+\overline{B}+C)}+\overline{(A+B+\overline{C})}}$$

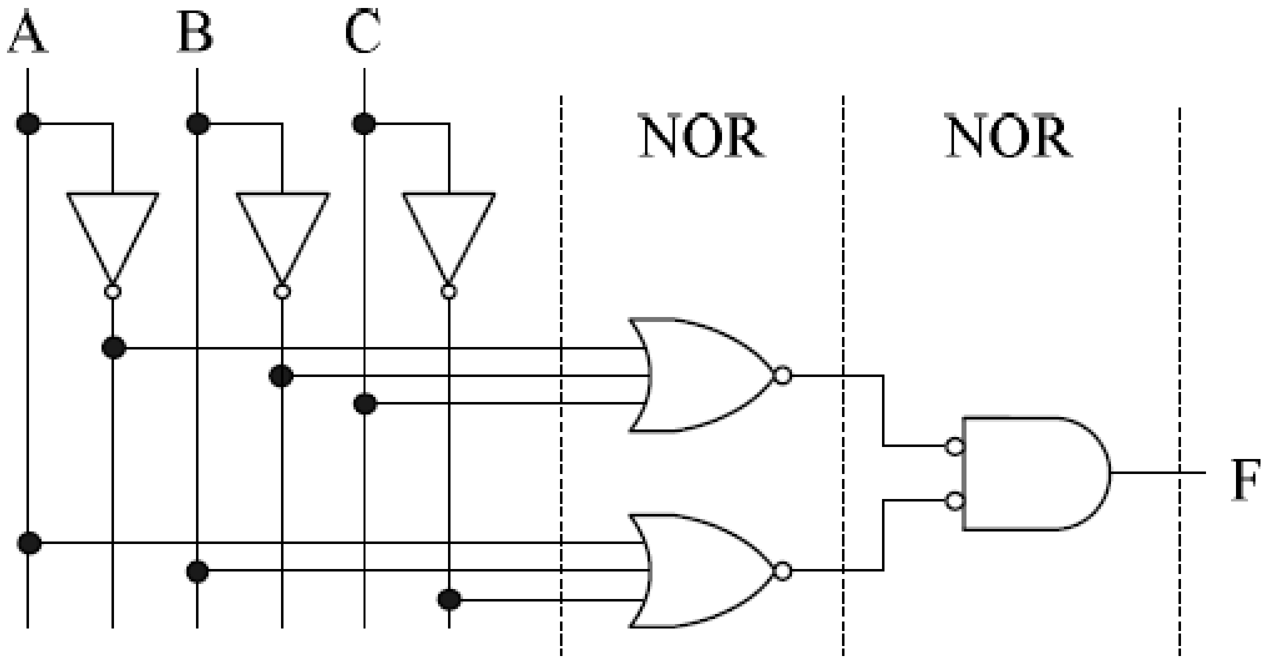


NOR-NOR circuit

Conversion of Circuits

$$F(A,B,C) = (\overline{A}+\overline{B}+C).(A+B+\overline{C}) = \overline{\overline{(\overline{A}+\overline{B}+C).(A+B+\overline{C})}}$$

$$= \overline{\overline{(\overline{A}+\overline{B}+C)}+\overline{(A+B+\overline{C})}}$$



NOR-NOR circuit

Conversion of Circuits

Example : $F(A,B,C) = A.\overline{B} + B.C + \overline{A}.\overline{C}$

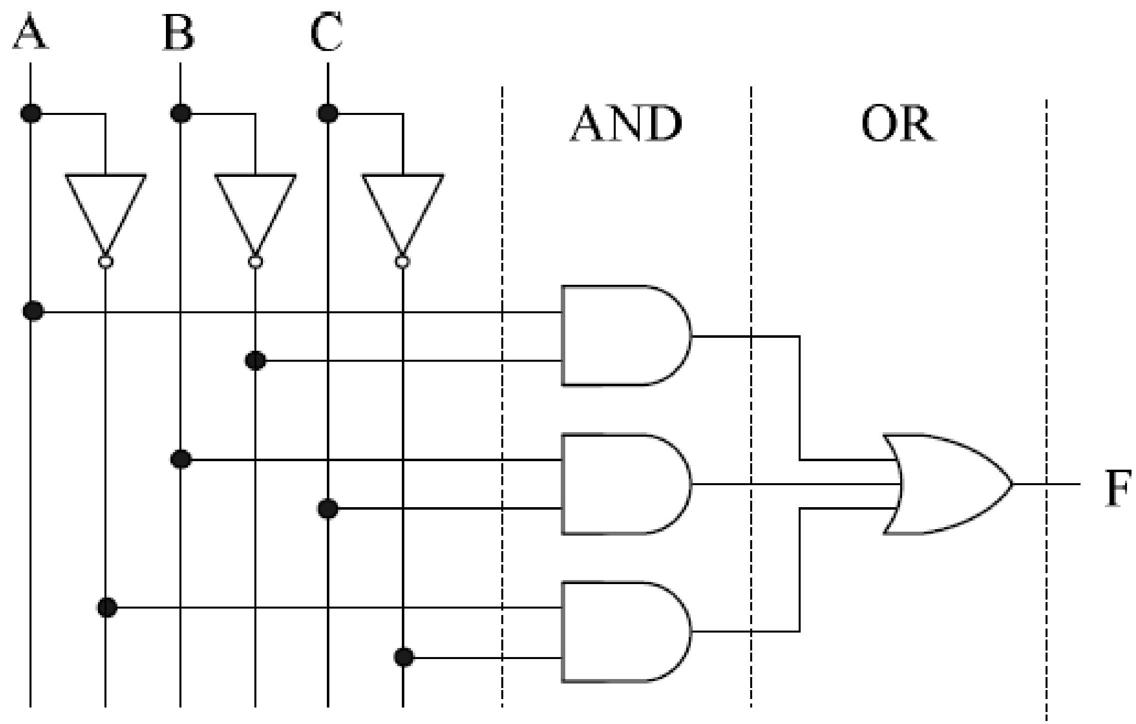
This SOP function can be implemented using:

i. AND-OR circuit

ii. NAND-NAND circuit

Conversion of Circuits

$$F(A, B, C) = A.\overline{B} + B.C + \overline{A}.\overline{C}$$



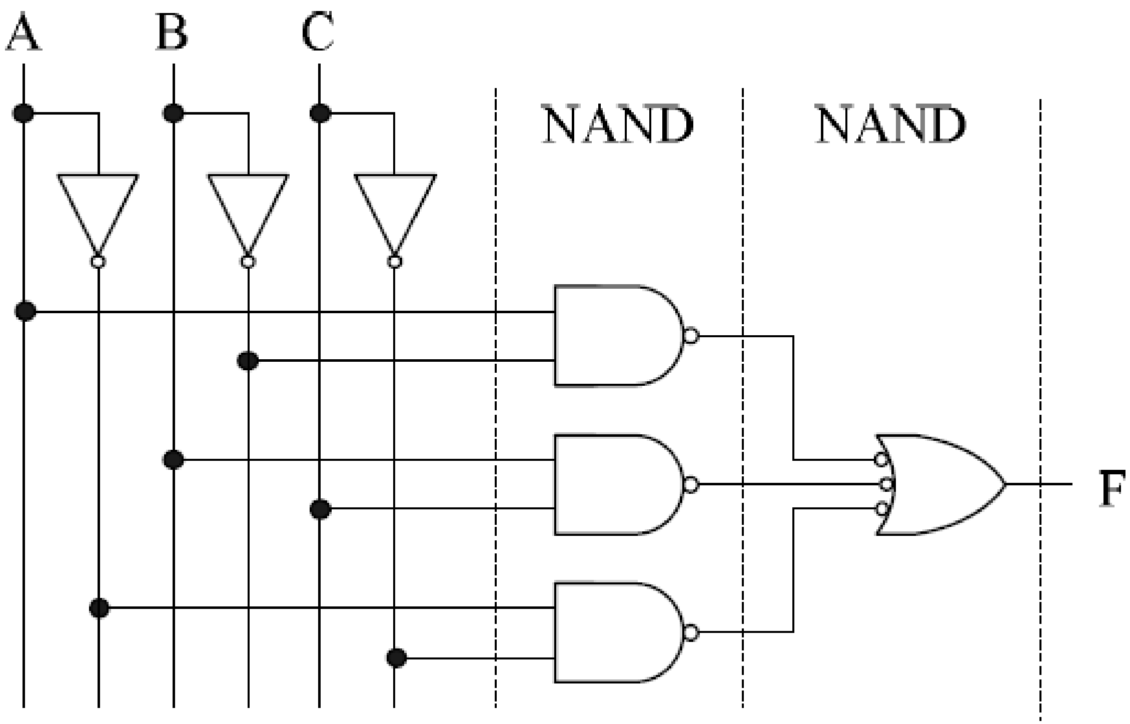
AND-OR circuit

Conversion of Circuits

$$F(A,B,C) = A.\bar{B} + B.C + \bar{A}.\bar{C} = \overline{\overline{A.\bar{B} + B.C + \bar{A}.\bar{C}}}$$

$$= \overline{(\overline{A.\bar{B}}).(\overline{B.C}).(\overline{\bar{A}.\bar{C}})}$$

Use DeMorgan's Theorem



NAND-NAND circuit

Conversion of Circuits

Example: $F(A,B,C) = (\overline{A} + \overline{B} + C).(A + B + \overline{C})$

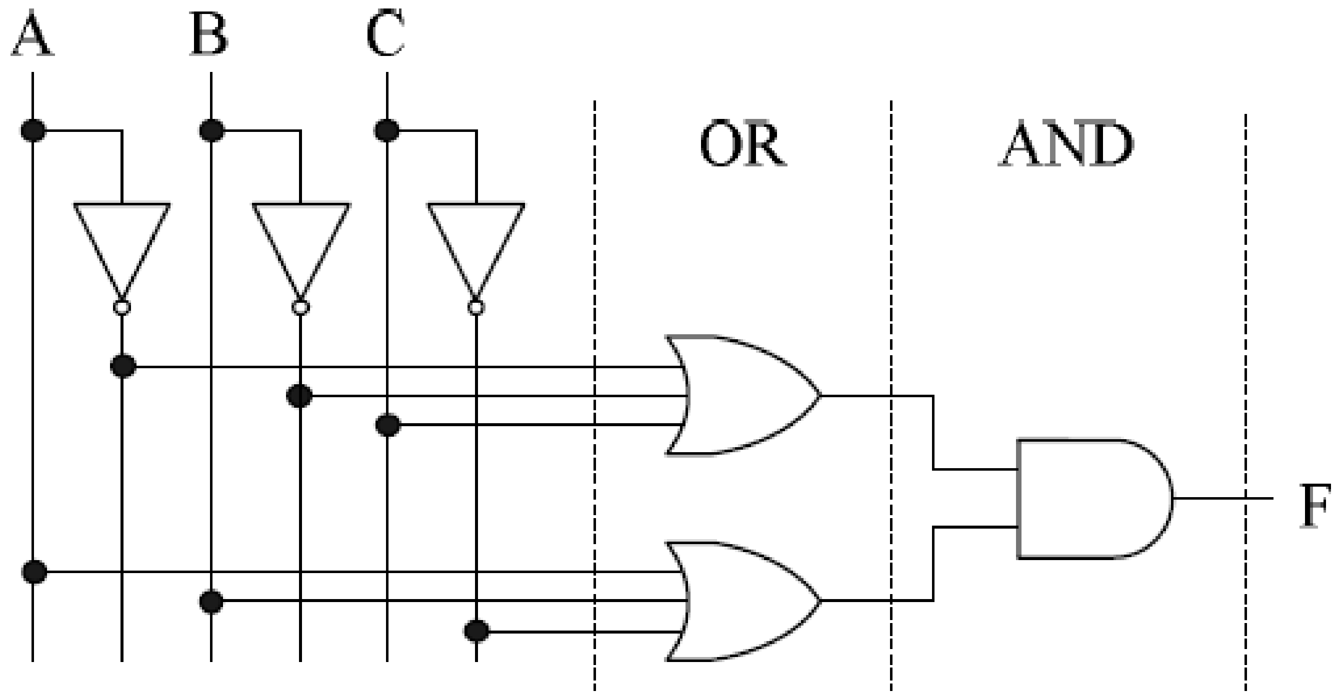
This POS function can be implemented using:

i. OR-AND circuit

ii. NOR-NOR circuit

Conversion of Circuits

$$F(A,B,C) = (\overline{A} + \overline{B} + C).(A + B + \overline{C})$$

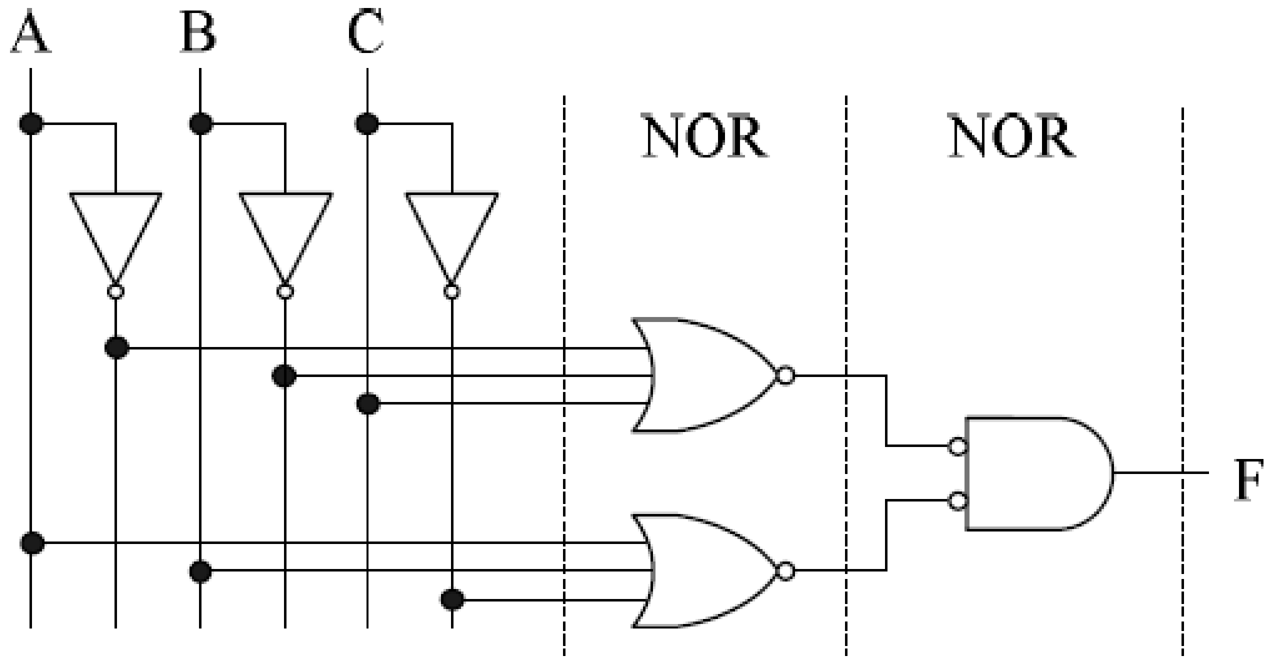


OR-AND circuit

Conversion of Circuits

$$F(A,B,C) = (\overline{A}+\overline{B}+C).(A+B+\overline{C}) = \overline{\overline{(\overline{A}+\overline{B}+C).(A+B+\overline{C})}}$$

$$= \overline{\overline{(\overline{A}+\overline{B}+C)}+\overline{(A+B+\overline{C})}}$$



NOR-NOR circuit